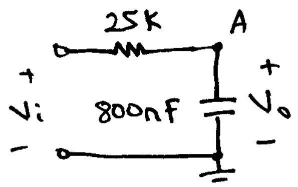
EGR 261
Signals and Systems
File: N261H4S

Solution to Homework Assignment #4

Problem Assignment:

1) For the circuit shown in Problem 13.49a in the Nilsson text:
   - A) Determine the differential equation that represents the system if $V_i(t)$ is the input voltage and $V_o(t)$ is the output voltage.
   - B) Find the unit step response, USR, using the differential equation. Also sketch the USR.
   - C) Find the impulse response, h(t), from the USR using h(t) = d/dt(USR). Also sketch h(t).
   - D) Find the transfer function $H(s) = V_o(s)/V_i(s)$ from the DE in part A
   - E) Find the unit step response, USR, from H(s)
   - F) Find the impulse response, h(t), from H(s)

A)



KCL, node A (out = +):

$$\frac{V_o - V_i}{25K} + 800 \times 10^{-9}\frac{dV_o}{dt} = 0$$

$$\boxed{\frac{dV_o}{dt} + 50\,V_o(t) = 50\,V_i(t)} \quad t \geq 0$$

B) $V_i(t) = u(t) = 1$ for $t \geq 0$, so $\dfrac{dV_o}{dt} + 50\,V_o(t) = 50$

Natural response:

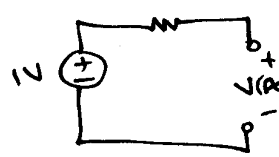
$s + 50 = 0$ (char. eq.) so $V_N = Ae^{-50t}$

Forced response:

$V_F = B$ (constant) so $V_F' = 0 \Rightarrow$ sub into D.E.: $0 + 50B = 50$

so $B = V_F = 1$

(or draw ckt at $t = \infty$



$V(\infty) = V_F = 1V$

Initial conditions:

$\boxed{t = 0^-} \rightarrow$ Dead circuit so $V_o(0^-) = 0 = V_o(0^+)$

Total response:

$V_o(t) = V_N + V_F = Ae^{-50t} + 1$

$V_o(0) = 0 = A + 1$, so $A = -1$

$$\boxed{USR = V_o(t) = (1 - e^{-50t})\,u(t) \quad V}$$

c) $h(t) = \frac{\partial}{\partial t}[uSR] = \frac{\partial}{\partial t}\left[(1 - e^{-50t})u(t)\right]$

$$= (1 - e^{-50t})\delta(t) + 50e^{-50t}u(t)$$

$$= (1 - e^{0})\delta(t) + 50e^{-50t}u(t)$$

$$\boxed{h(t) = 50e^{-50t}u(t)}$$

D) $\frac{\partial V_o}{\partial t} + 50V_o(t) = 50V_i(t)$

$\mathcal{L}$

$$\left[sV_o(s) - \underbrace{V_o(0)}_{0}\right] + 50V_o(s) = 50V_i(s)$$

$$V_o(s)\left[s + 50\right] = 50V_i(s)$$

$$\boxed{H(s) = \frac{V_o(s)}{V_i(s)} = \frac{50}{s+50}}$$

E) $uSR = \mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{50}{s(s+50)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+50}\right\}$

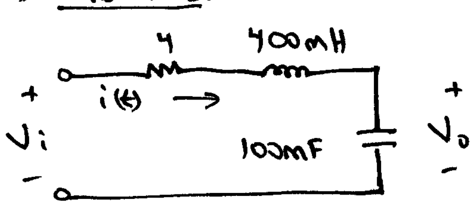
$uSR = u(t) - e^{-50t}u(t)$ $\quad \boxed{uSR = (1 - e^{-50t})u(t)}$

F) $h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{50}{s+50}\right\}$

$$\boxed{h(t) = 50e^{-50t}u(t)}$$

2) Repeat problem 1 for the circuit shown in Problem 13.71 in the Nilsson text.

A) Find D.E.



KVL: $-V_i + 4i(t) + 0.4\frac{\partial i}{\partial t} + V_o = 0$

Capacitor: $i(t) = 0.1\frac{\partial V_o}{\partial t}$ so $\frac{\partial i}{\partial t} = 0.1\frac{\partial^2 V_o}{\partial t^2}$

(sub for $i(t)$ + $\frac{\partial i}{\partial t}$ in KVL)

$$-V_i + 4\left[0.1\frac{\partial V_o}{\partial t}\right] + 0.4\left[0.1\frac{\partial^2 V_o}{\partial t^2}\right] + V_o = 0$$

$$\boxed{\frac{\partial^2 V_o}{\partial t^2} + 10\frac{\partial V_o}{\partial t} + 25V_o(t) = 25V_i(t)} \text{ D.E. } (t \geq 0)$$

B) Find the USR:

$V_i(t) = u(t) = 1 \quad (\text{for } t \geq 0)$

so $\dfrac{\partial^2 V_o}{\partial t^2} + 10\dfrac{\partial V_o}{\partial t} + 25V_o(t) = 25$

natural response:

char. eq $\rightarrow s^2 + 10s + 25 = 0$

$(s+5)^2 = 0 \quad \rightarrow \quad s_1 = s_2 = -5$ (critically damped)

$V_N = (A_1 t + A_2)e^{-5t}$

forced response: $V_F = V(\infty)$

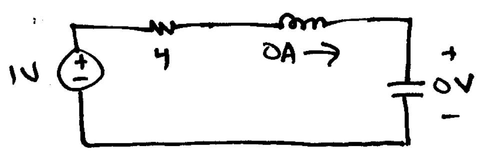
$\boxed{t = \infty}$ steady state

$V(\infty) = 1 = V_F$

Initial Conditions

$\boxed{t = 0^-}$ Dead ckt so $V_o(0^-) = 0$, $i_L(0^-) = 0$

$\boxed{t = 0^+}$ $V_o(0^+) = V_o(0^-) = 0$, $i_L(0^+) = i_L(0^-) = 0$



$i(t) = C\dfrac{\partial V_o(t)}{\partial t}$

so

$\dfrac{\partial V_o(0^+)}{\partial t} = \dfrac{i_c(0^+)}{C} = 0$

total response

$V_o(t) = V_N + V_F$

$V_o(t) = (A_1 t + A_2)e^{-5t} + 1$

Find A_1 + A_2 using initial conditions ($V_o(0) = 0$, $V_o'(0) = 0$)

$0 = A_2 + 1 \quad \rightarrow \quad A_2 = -1$

$\dfrac{\partial}{\partial t}$

$\dfrac{\partial V_o(t)}{\partial t} = A_1 e^{-5t} + (A_1 t + A_2)(-5e^{-5t}) + 0$

$\dfrac{\partial V_o(0)}{\partial t} = 0 = A_1 - 5A_2 = A_1 + 5 \quad \rightarrow A_1 = -5$

so $\boxed{V_o(t) = \left[1 - (5t+1)e^{-5t}\right]u(t) = \text{USR}}$

c) Find the impulse response:

$$h(t) = \frac{\partial}{\partial t}\left[USR\right] = \frac{\partial}{\partial t}\left[\left[1-(5t+1)e^{-5t}\right]u(t)\right]$$

$$h(t) = \left[1-(5t+1)e^{-5t}\right]\delta(t) \; \left[-5e^{-5t} + 5(5t+1)e^{-5t}\right]u(t)$$

$$h(t) = \left[1-(0+1)e^{0}\right]\delta(t) + 25te^{-5t}u(t)$$

$$\boxed{h(t) = 25te^{-5t}u(t) = \text{impulse response}}$$

D) Find H(s) from D.E.

$$\frac{\partial^2 V_o}{\partial t^2} + 10\frac{\partial V_o}{\partial t} + 25V_o(t) = 25V_i(t)$$

$$\mathcal{L}\quad \left[s^2V_o(s) - s\underbrace{V_o(0)}_{0} - \underbrace{V_o'(0)}_{0}\right] + 10\left[sV_o(s) - s\underbrace{V_o(0)}_{0}\right] + 25V_o(s) = 25V_i(s)$$

$$V_o(s)\left[s^2+10s+25\right] = 25V_i(s)$$

$$\boxed{H(s) = \frac{V_o(s)}{V_i(s)} = \frac{25}{(s+5)^2}}$$

E) Find USR from H(s):

$$USR = \mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\}$$

$$\frac{H(s)}{s} = \frac{25}{s(s+5)^2} = \frac{A}{s} + \frac{B}{(s+5)^2} + \frac{C}{s+5} = \frac{1}{s} - \frac{5}{(s+5)^2} - \frac{1}{s+5}$$

$$\text{so}\quad USR = u(t) - 5te^{-5t}u(t) - e^{-5t}u(t)$$

$$\boxed{USR = \left[1-(5t+1)e^{-5t}\right]u(t)}$$

F) Find h(t) from H(s)

$$\boxed{h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{25}{(s+5)^2}\right\} = 25te^{-5t}u(t) = \text{impulse response}}$$

For problems 3, 4, 5, and 6 below use a graphical approach in evaluating the convolution integral. Include sketches to illustrate each part of the solution as well as the related calculations. Sketch the final result as well.

3) Problem 2.4-18, parts a,c,f, and g in the Lathi text (Hint: Invert x_1 rather than x_2 on part g.) Find and sketch c(t) = x1(t)*x2(t) for the waveforms below.

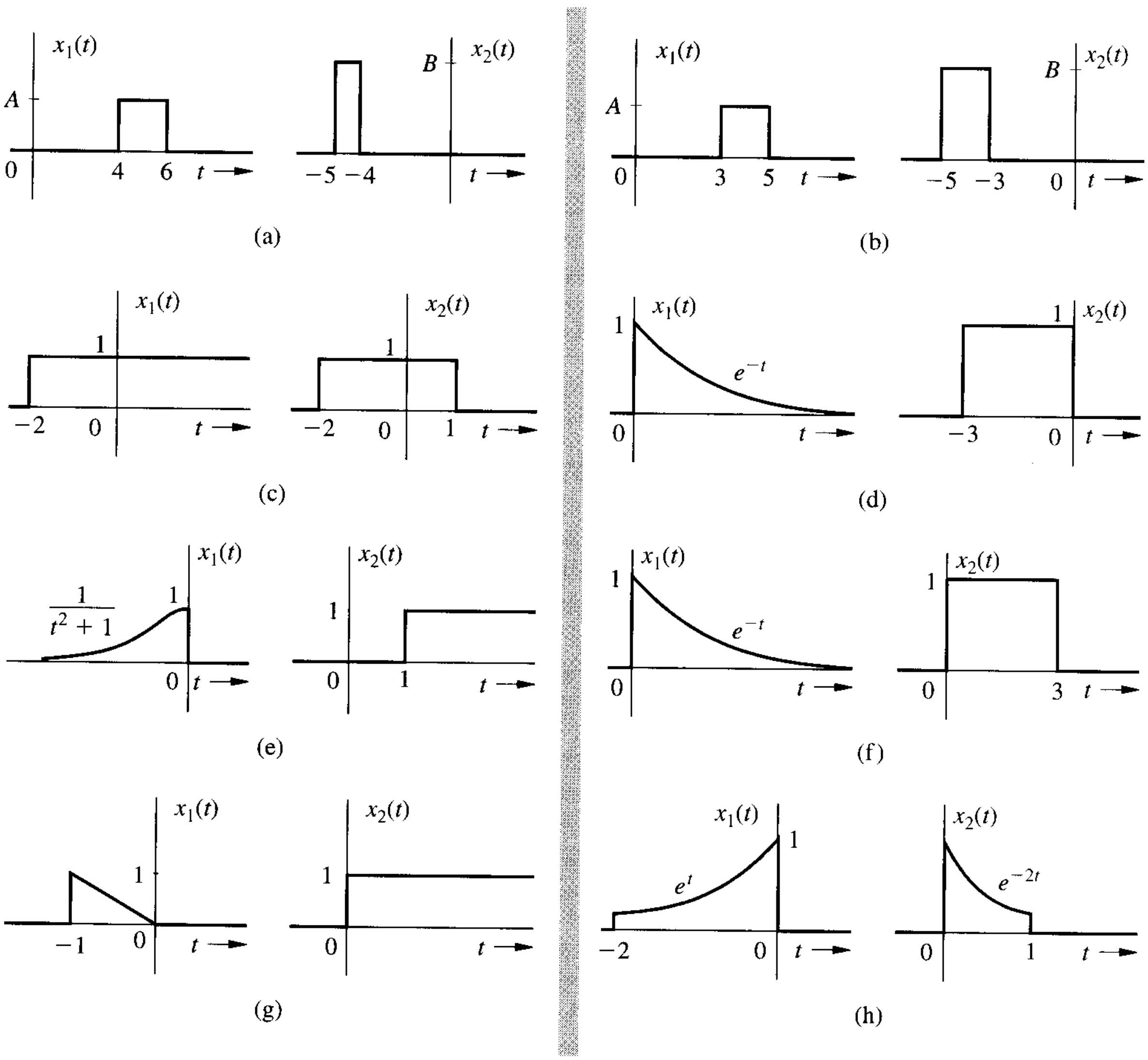
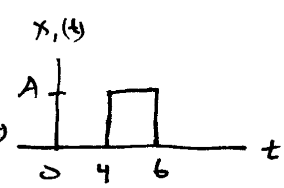
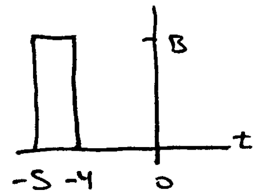


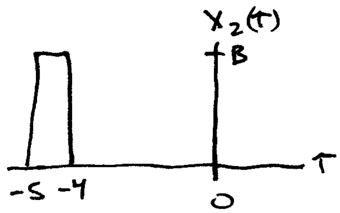
Figure P2.4-18

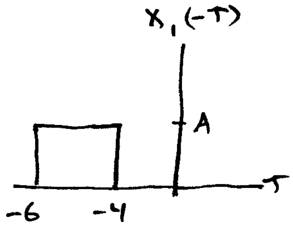
2.4-18a) Find $x_1(t) * x_2(t) = c(t)$

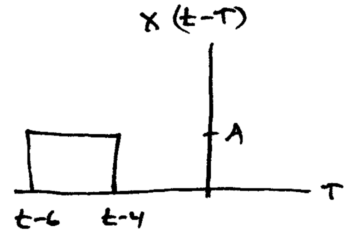




Solution:

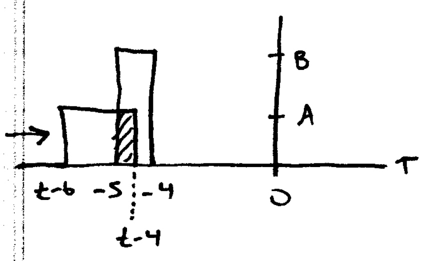






A) $t < -1 \rightarrow$ no overlap so $\boxed{c(t) = 0}$ $(t < -1)$

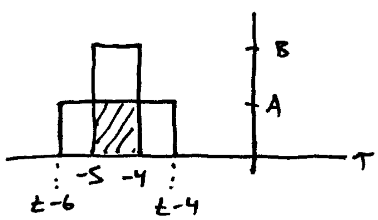
B) $-1 \le t \le 0$



$$c(t) = \int_{-5}^{t-4} AB\,d\tau = AB\tau \Big|_{-5}^{t-4} = AB(t-4) - AB(-5)$$

$$\boxed{c(t) = AB(t+1)} \quad (-1 \le t \le 0)$$

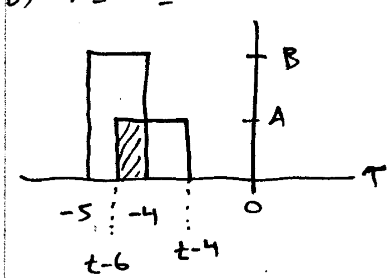
C) $0 \le t \le 1$



$$c(t) = \int_{-5}^{-4} AB\,d\tau = AB\tau \Big|_{-5}^{-4} = AB(-4) - AB(-5)$$

$$\boxed{c(t) = AB} \quad (0 \le t \le 1)$$

D) $1 \le t \le 2$



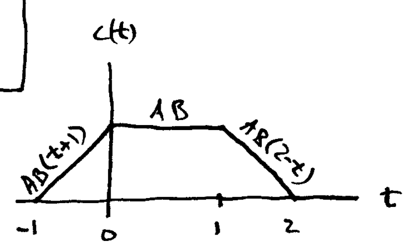
$$c(t) = \int_{t-6}^{-4} AB\,d\tau = AB\tau \Big|_{t-6}^{-4} = AB(-4) - AB(t-6)$$

$$\boxed{c(t) = AB(2-t)} \quad (1 \le t \le 2)$$

E) $t \ge 2 \rightarrow$ no overlap so $c(t) = 0$

Summary:

$$c(t) = \begin{cases} 0 & t < -1 \\ AB(t+1) & -1 \le t \le 0 \\ AB & 0 \le t \le 1 \\ AB(2-t) & 1 \le t \le 2 \\ 0 & t \ge 2 \end{cases}$$



(b)

$c(t) = \int_{1.5+t}^{2.5} AB\,d\tau = AB(1-t) \qquad 0 \le t \le 1$

$c(t) = \int_{1.5}^{2.5+t} AB\,d\tau = AB(t+1) \qquad -1 \le t \le 0$

$c(t) = 0 \qquad \text{for} \quad |t| \ge 1$

(c)

$c(t) = \int_{-1+t}^{2+t} d\tau = 3 \qquad t > -1$

$c(t) = \int_{-2}^{2+t} d\tau = t+4 \qquad -1 \ge t \ge -4$

$c(t) = 0 \qquad t \le -4$

(d)

$$\begin{aligned} c(t) &= \int_t^{3+t} e^{-\tau}\,d\tau = e^{-t}(1-e^{-3}) = 0.95e^{-t} \qquad t \ge 0 \\ &= \int_0^{3+t} e^{-\tau}\,d\tau = 1-e^{-(3+t)} = 1-0.0498e^{-t} \qquad 0 \ge t \ge -3 \\ &= 0 \qquad t \le -3 \end{aligned}$$

(e)

$c(t) = \int_{-\infty}^{-1+t} \frac{1}{\tau^2+1}\,d\tau = \tan^{-1}(t-1) + \frac{\pi}{2} \quad t \le 1$

$c(t) = \int_{-\infty}^{0} \frac{1}{\tau^2+1}\,d\tau = \tan^{-1}\tau\Big|_{-\infty}^{0} = \frac{\pi}{2} \quad t \ge 1$

(f)

$c(t) = \int_0^t e^{-\tau}\,d\tau = 1-e^{-t} \qquad 0 \le t \le 3$

$c(t) = \int_{t-3}^{t} e^{-\tau}\,d\tau = e^{-(t-3)} - e^{-t} \qquad t \ge 3$

$c(t) = 0 \qquad t \le 0$

(g) This problem is more conveniently solved by inverting $x_1(t)$ rather than $x_2(t)$

$c(t) = \int_t^{t+1} (\tau - t)\,d\tau = \frac{1}{2} \qquad t \ge 0$

$c(t) = \int_0^{t+1} (\tau - t)\,d\tau = \frac{1}{2}(1-t^2) \qquad 0 \ge t \ge -1$

$c(t) = 0 \qquad \text{for}$

$t \le -1$

(h) $x_1(t) = e^t, \quad x_2(t) = e^{-2t}, \quad x_1(\tau) = e^\tau, \quad x_2(t-\tau) = e^{-2(t-\tau)}.$

$$\begin{aligned} c(t) &= \int_{-1+t}^{0} e^\tau e^{-2(t-\tau)}\,d\tau = e^{-2t}\int_{-1+t}^{0} e^{3\tau}\,d\tau = \frac{1}{3}[e^{-2t} - e^{t-3}] \qquad 0 \le t \le 1 \\ c(t) &= \int_{-1+t}^{t} e^\tau e^{-2(t-\tau)}\,d\tau = e^{-2t}\int_{-1+t}^{t} e^{3\tau}\,d\tau = \frac{1}{3}[e^{t} - e^{t-3}] \qquad 0 \ge t \ge -1 \\ c(t) &= \int_{-2}^{t} e^\tau e^{-2(t-\tau)}\,d\tau = e^{-2t}\int_{-2}^{t} e^{3\tau}\,d\tau = \frac{1}{3}[e^{t} - e^{-2(t+3)}] \qquad -1 \ge t \ge -2 \\ c(t) &= 0 \qquad t \le -2 \end{aligned}$$

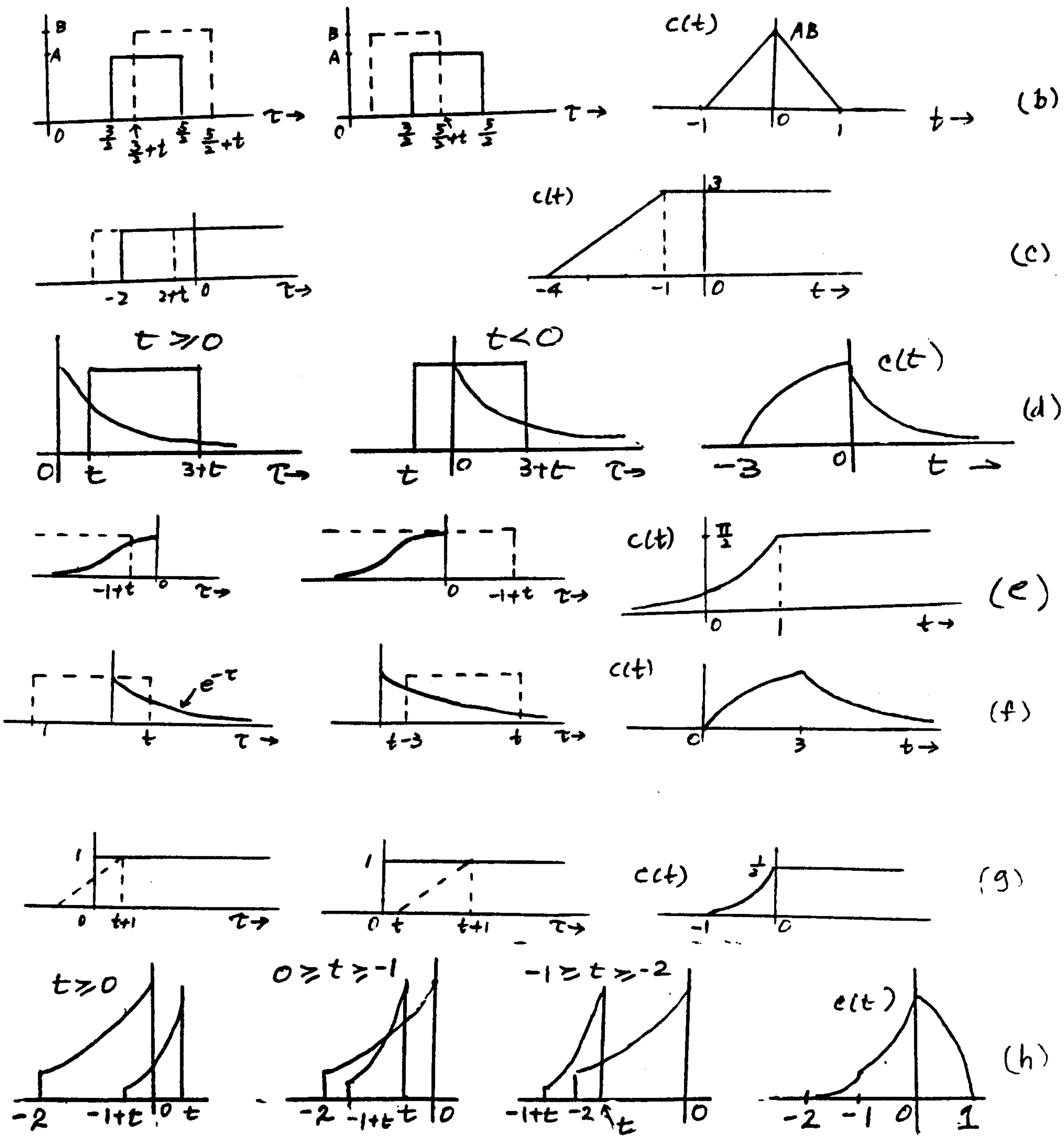
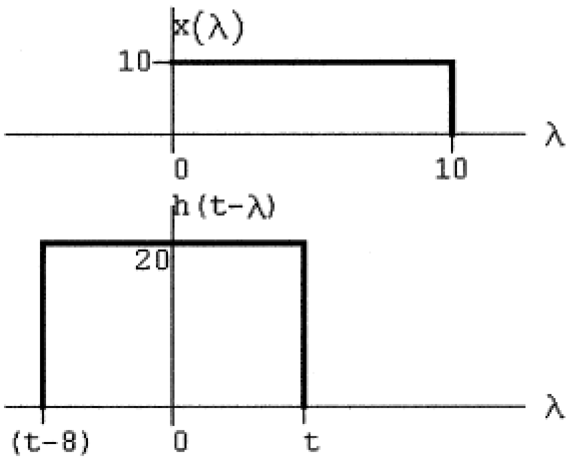


Fig. S2.4-18

4) Problem 13.59b in the Nilsson text



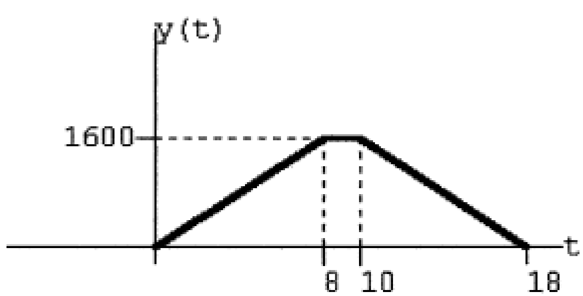
$$y(t) = 0 \qquad t < 0$$

$$0 \le t \le 8: \qquad y(t) = \int_0^t 200\,d\lambda = 200t$$

$$8 \le t \le 10: \qquad y(t) = \int_{t-8}^t 200\,d\lambda = 200(t - t + 8) = 1600$$

$$10 \le t \le 18: \qquad y(t) = \int_{t-8}^{10} 200\,d\lambda = 200(18 - t)$$

$$18 \le t < \infty: \qquad y(t) = 0$$



5) Problem 13.61 in the Nilsson text

P 13.61 **[a]** $-1 \leq t \leq 4$:

$$v_o = 20\int_0^{t+1} 3\lambda\,d\lambda = 30\lambda^2 \Big|_0^{t+1} = 30t^2 + 60t + 30$$

$4 \leq t \leq 7$:

$$v_o = 20\int_0^5 3\lambda\,d\lambda + 20\int_5^{t+1} (20-\lambda)\,d\lambda$$

$$= 30\lambda^2 \Big|_0^5 + 400\lambda \Big|_5^{t+1} - 10\lambda^2 \Big|_5^{t+1}$$

$$= -10t^2 + 380t - 610$$

$7 \leq t \leq 12$:

$$v_o = 20\int_{t-7}^5 3\lambda\,d\lambda + 20\int_5^{t+1} (20-\lambda)\,d\lambda$$

$$= 30\lambda^2 \Big|_{t-7}^5 + 400\lambda \Big|_5^{t+1} - 10\lambda^2 \Big|_5^{t+1}$$

$$= -40t^2 + 800t - 2080$$

$12 \leq t \leq 19$:

$$v_o = 20\int_{t-7}^{t+1} (20-\lambda)\,d\lambda = 400\lambda \Big|_{t-7}^{t+1} - 10\lambda^2 \Big|_{t-7}^{t+1}$$

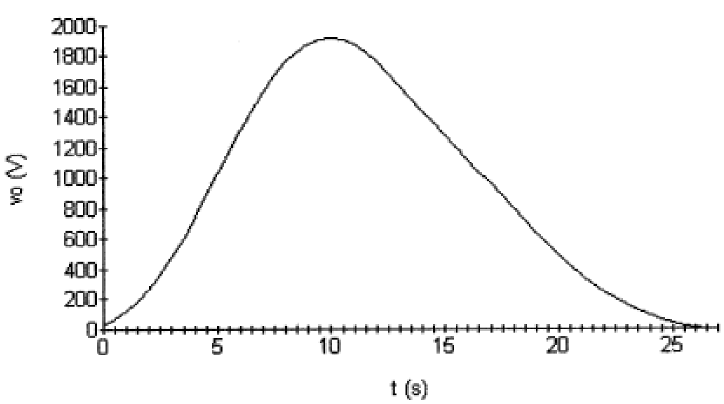
$$= -160t + 3680$$

$19 \leq t \leq 27$:

$$v_o = 20\int_{t-7}^{20} (20-\lambda)\,d\lambda = 400\lambda \Big|_{t-7}^{20} - 10\lambda^2 \Big|_{t-7}^{20}$$

$$= 10t^2 - 540t + 7290$$

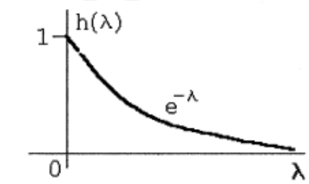
[b]

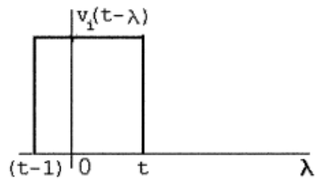


6) Problem 13.57 in the Nilsson text

P 13.57 $H(s) = \dfrac{V_o}{V_i} = \dfrac{1}{s+1}$; $\qquad h(t) = e^{-t}$

For $\quad 0 \le t \le 1$:





$$v_o = \int_0^t e^{-\lambda}\,d\lambda = (1 - e^{-t})\,\mathrm{V}$$

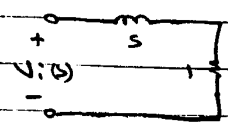
For $\quad 1 \le t \le \infty$:

$$v_o = \int_{t-1}^t e^{-\lambda}\,d\lambda = (e-1)e^{-t}\,\mathrm{V}$$

7) Repeat problem 13.57 using Laplace transforms. Specifically, find $H(s) = V_o(s)/V_i(s)$ and then find $v_o(t) = \mathcal{L}^{-1}\{H(s)V_i(s)\}$.

$v_i = [u(t) - u(t-1)]$ so $V_i(s) = \dfrac{1}{s} - \dfrac{e^{-s}}{s} = \dfrac{1-e^{-s}}{s}$

S-Domain ckt:



$V_o(s) = V_i(s)\left[\dfrac{1}{s+1}\right]$ so $\boxed{H(s) = \dfrac{V_o(s)}{V_i(s)} = \dfrac{1}{s+1}}$

(Note: $h(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\} = e^{-t}u(t)$)

$V_o(s) = H(s)\cdot V_i(s)$

$$V_o(s) = \left(\frac{1}{s+1}\right)\left[\frac{1}{s} - \frac{e^{-s}}{s}\right] = \underbrace{\frac{1}{s(s+1)}}_{V_1} - \underbrace{\frac{e^{-s}}{s(s+1)}}_{-V_1 e^{-s} = V_2}$$

$$V_1(s) = \frac{1}{s(s+1)} = \frac{1}{s} + \frac{-1}{s+1}$$

so $v_1(t) = u(t) - e^{-t}u(t) = (1-e^{-t})u(t)$

$v_2(t)$ is a delayed (+ negated) version of $v_1(t)$, so

$v_2(t) = -(1-e^{-(t-1)})u(t-1)$

$v_o(t) = v_1(t) + v_2(t)$

so $\boxed{v_o(t) = (1-e^{-t})u(t) - (1-e^{-(t-1)})u(t-1)}$

or
$$v_o(t) = \begin{cases} 0 & t<0 \\ 1-e^{-t} & 0 \le t < 1 \\ (1-e^{-t}) - (1-e^{-(t-1)}) = -e^{-t} + e^{-(t-1)} & t \ge 1 \end{cases}$$